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Note.—For this same problem, see Byerly's *Integral Calculus*, Chap. XVI, Elliptic Integrals. In the 1902 edition, however, an incorrect answer is given.

Also solved by H. S. Uhler.

MECHANICS.

307. Proposed by LAENAS G. WELD, Pullman, Illinois.

Four forces W, X, Y, and Z, concurrent in O, are in equilibrium. Prove that

$$W : X : Y : Z :: \Delta_1 : \Delta_2 : \Delta_3 : \Delta_4,$$

where

$$\Delta_1 = \begin{vmatrix} 1 & \cos XOY & \cos XOZ \\ \cos XOY & 1 & \cos YOZ \\ \cos XOZ & \cos YOZ & 1 \end{vmatrix}^{\frac{1}{2}}; \quad \Delta_2 = \begin{vmatrix} 1 & \cos WOY & \cos WOZ \\ \cos WOY & 1 & \cos YOZ \\ \cos WOZ & \cos YOZ & 1 \end{vmatrix}^{\frac{1}{2}};$$

$$\Delta_3 = \begin{vmatrix} 1 & \cos WOX & \cos WOZ \\ \cos WOX & 1 & \cos XOZ \\ \cos WOZ & \cos XOZ & 1 \end{vmatrix}^{\frac{1}{2}}; \quad \Delta_4 = \begin{vmatrix} 1 & \cos WOX & \cos WOY \\ \cos WOX & 1 & \cos XOY \\ \cos WOY & \cos XOY & 1 \end{vmatrix}^{\frac{1}{2}}.$$

Solution by H. S. Uhler, Yale University.

For brevity let

$$l \equiv \cos WOX = \cos XOW, \quad p \equiv \cos XOY = \cos YOX,$$
$$m \equiv \cos WOY = \cos YOW, \quad q \equiv \cos XOZ = \cos ZOX,$$
$$n \equiv \cos WOZ = \cos ZOW, \quad r \equiv \cos YOZ = \cos ZOY.$$

Since the forces are in equilibrium the (algebraic) sum of their projections on any straight line must vanish. Hence, by projecting the forces successively upon the lines of action of W, X, Y, and Z, respectively, we obtain the following redundant set of homogeneous equations:

$$W + lX + mY + nZ = 0,$$
$$lW + X + pY + qZ = 0,$$
$$mW + pX + Y + rZ = 0,$$
$$nW + qX + rY + Z = 0.$$

By applying a well-known theorem of determinants to the cofactors of the first row we find

$$W : X :: \begin{vmatrix} 1 & p & q \\ p & 1 & r \\ q & r & 1 \end{vmatrix} : - \begin{vmatrix} l & p & q \\ m & 1 & r \\ n & r & 1 \end{vmatrix}, \quad \text{or} \quad W : X :: -\Delta_1^2 : \begin{vmatrix} l & p & q \\ m & 1 & r \\ n & r & 1 \end{vmatrix}. \tag{1}$$

Similarly, for the second line or row

$$W : X :: \begin{vmatrix} l & m & n \\ p & 1 & r \\ q & r & 1 \end{vmatrix} : - \begin{vmatrix} 1 & m & n \\ m & 1 & r \\ n & r & 1 \end{vmatrix}, \quad \text{or} \quad W : X :: \begin{vmatrix} l & m & n \\ p & 1 & r \\ q & r & 1 \end{vmatrix} : -\Delta_2^2. \tag{2}$$

The determinants constituting the fourth and third terms of proportions (1) and (2), respectively, are equal because the rows of one are the same as the corresponding columns of the other, hence the product of (1) and (2) is

$$W^2 : X^2 :: \Delta_1^2 : \Delta_2^2,$$

or, since we are only dealing with the arithmetical magnitudes of the forces,

$$W : X :: \Delta_1 : \Delta_2.$$

Obviously cyclical permutation of the symbols leads to the required proportions

$$W : X : Y : Z :: \Delta_1 : \Delta_2 : \Delta_3 : \Delta_4.$$

Remark: From the principles of statics we know that the four original scalar equations must be compatible or simultaneous and, from the theory of determinants, that the necessary and sufficient condition for this compatibility is the vanishing of the determinant Δ of the coefficients of W, X, Y, and Z. Hence, the six cosines l, m, n, p, q, r pertaining to a closed space quadrilateral are not all independent but are mutually connected by the relation $\Delta = 0$, that is,

$$l^2 + m^2 + n^2 + p^2 + q^2 + r^2 - l^2r^2 - m^2q^2 - n^2p^2 - 2lmp$$
$$- 2lnq - 2mnr - 2pqr + 2lmqr + 2lnpr + 2mnpq = 1.$$

310. Proposed by EMMA M. GIBSON, Drury College.

"A particle moveable on a smooth spherical surface of radius a is projected along the horizontal great circle with a velocity v which is great compared with $\sqrt{(2ga)}$. Prove that its path lies between this great circle and a parallel circle whose plane is at a depth $2ga^2/v^2$ below the centre, approximately."

From Lamb's *Dynamics*, p. 334, Ex. 3.

SOLUTION BY H. S. UHLER, Yale University.

Since the particle has only two degrees of freedom let us choose as coördinates the angles θ and ϕ which the radius terminating in the particle (of mass m) makes with any fixed vertical plane which contains the center of the sphere and with the horizontal plane passing through this center, respectively. Since none of the forces acting on the particle has a moment about the vertical diameter of the sphere it follows that the moment of momentum about this axis must remain invariable. At the instant of projection the moment of momentum equals $a \cdot mv$. At some later time the moment of momentum will be $r \cdot mr\dot{\theta}$ or $ma^2(\cos^2 \phi)\dot{\theta}$, because $r = a \cos \phi$. Consequently

$$a(\cos^2 \phi)\dot{\theta} = v. \tag{1}$$

Another relation between θ and ϕ may be derived from the principle of the conservation of energy. At the instant of projection the total energy equals $\frac{1}{2}mv^2 + V$, where V symbolizes the potential energy at the level of the center of the sphere. In general, at any later time the components of the kinetic energy in the horizontal and vertical planes will be respectively $\frac{1}{2}ma^2(\cos^2 \phi)\dot{\theta}^2$ and $\frac{1}{2}ma^2\dot{\phi}^2$, while the potential energy will be $V - mga \sin \phi$. Therefore,

$$a^2(\cos^2 \phi)\dot{\theta}^2 + a^2\dot{\phi}^2 - 2ag \sin \phi = v^2. \tag{2}$$

Substituting the value of $\dot{\theta}$ from (1) in (2) we find

$$a^2\dot{\phi}^2 = 2ag \sin \phi - v^2 \tan^2 \phi$$

or

$$a\dot{\phi} = (2ag \sin \phi - v^2 \tan^2 \phi)^{\frac{1}{2}}.$$

The last equation shows that ϕ cannot be negative with $\dot{\phi}$ real, in other words, the particle will not rise above the plane of projection. Writing this equation in the form

$$2a\sqrt{2ag}(\cos \phi)\dot{\phi} = [(\sin \phi)(\sqrt{v^4 + 16a^2g^2} + v^2 + 4ag \sin \phi)(\sqrt{v^4 + 16a^2g^2} - v^2 - 4ag \sin \phi)]^{\frac{1}{2}},$$

it becomes self-evident that the greatest value which ϕ can attain without making $\dot{\phi}$ complex is given by the formula

$$4ag \sin \phi_0 = \sqrt{v^4 + 16a^2g^2} - v^2.$$

The corresponding vertical distance below the horizontal diametral plane is $a \sin \phi_0 = d$. Then, rigorously,

$$d = \frac{1}{4g}(\sqrt{v^4 + 16a^2g^2} - v^2).$$

When $4ag/v^2$ is less than unity we may expand the radical by the binomial theorem to obtain an approximate rational expression for d. Then

$$d = \frac{v^2}{4g}\left[\left(1 + \frac{8a^2g^2}{v^4} - \frac{32a^4g^4}{v^8} + \cdots\right) - 1\right]$$